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Vector Spaces and Subspaces

1. State the definition of "V is a real *vector space*"

V is a set of elements that can be scaled & added

such that ① V is closed under scaling

② V is closed under adding

③ adding & scaling works like normal

2. Let $\mathbb{P}_2 = \{ \quad at^2 + bt + c \quad : \quad a, b, c \in \mathbb{R} \quad \}$. Show that $\mathbb{P}_2$ is closed under scaling and adding.

$$(at^2+bt+c)+(dt^2+et+f) = \underbrace{(a+d)t^2+(b+e)t+(c+f)}_{\text{still in } \mathbb{R}}$$

$$r(at^2+bt+c) = \underbrace{(ra)t^2+(rb)t+(rc)}_{\text{still in } \mathbb{R}.}$$

3. Let V be a vector space. State the definition of a "H is a *subspace* of V".

$H \subseteq V$ satisfies

① $\vec{0} \in H$

② H is closed under scaling

③ H is closed under adding

4. Complete the blank in the the following.

Theorem:

$H \subseteq V$ is a subspace of V

and only if

there is a set of vectors $\mathscr{U} \subset H$ so that $H =$ Span $\mathcal{U}$

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5. Let $H = \left\{ \begin{bmatrix} 2t+s \\ t-3s \\ t \end{bmatrix} \in \mathbb{R}^3 : s,t \in R \right\}$ be a subspace of $\mathbb{R}^3$. Prove that H is the span of a set of vectors by *rewriting the set-builder notation.* You must show *all steps.*

$$H = \left\{ \begin{bmatrix} 2t + s \\ t - 3s \\ t + 0 \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$= \left\{ t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$= \left\{ \vec{b} \in \mathbb{R}^3 : \vec{b} = t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \text{ for } s,t \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \right\}$$

6. Let $H = \left\{ \begin{bmatrix} 3a+2b \\ c \\ a-3b \\ b+2c \end{bmatrix} \in \mathbb{R}^4 : a,b,c \in R \right\}$ be a subspace of $\mathbb{R}^4$. Prove that H is the span of a set of vectors by *rewriting the set-builder notation.* You must show *all steps.*

$$H = \left\{ \begin{bmatrix} 3a + 2b + 0 \\ 0 + 0 + c \\ a - 3b + 0 \\ 0 + b + 2c \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$$

$$= \left\{ a\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$$

$$= \left\{ \vec{b} \in \mathbb{R}^4 : \vec{b} = a\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \text{ for } a,b,c \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

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7. Let

$$H = \left\{ \begin{bmatrix} 2p-3q \\ 4p \\ \underline{\underline{1+q}} \end{bmatrix} \in \mathbb{R}^3 : \ p, q \text{ are in } \mathbb{R} \right\}$$

Determine if H is a subspace of $\mathbb{R}^3$. If it is, write it as the span of a set of vectors.

<u>Claim</u>: H is <u>NOT</u> a subspace.

we will prove that $\vec{0} \notin H$.

<u>proof</u> (by computation)

$1+q=0 \Rightarrow q=-1$

$4p=0 \Rightarrow p=0$

<u>BUT</u> $p=0, q=-1 \Rightarrow 2p-3q \neq 0$

<u>so</u> $\vec{0} \notin H$

<u>so</u> H is <u>NOT</u> a subspace

<u>ALTERNATE</u> proof

$$H = \left\{ \begin{bmatrix} 2p-3q+0 \\ 4p+0+0 \\ 0+q+1 \end{bmatrix} : p,q \in \mathbb{R} \right\}$$

$$= \left\{ p\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + q\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : p,q \in \mathbb{R} \right\}$$

$$= \left\{ \vec{b} \in \mathbb{R}^3 : \vec{b} = p\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + q\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ p,q \in \mathbb{R} \right.$$

<u>NOTICE</u>

$\neq$ Span of any set of vectors

H is not a span

$\Rightarrow H$ is not a space

8. Let

$$H = \left\{ \begin{bmatrix} 3s+2t \\ t-3s \\ 2t+s \end{bmatrix} \in \mathbb{R}^3 : \ s, t \text{ are in } \mathbb{R} \right\}$$

Determine if H is a subspace of $\mathbb{R}^3$. If it is, write it as the span of a set of vectors.

<u>Claim</u>: H <u>IS</u> a subspace.

we will prove this by writing H as a span.

$$H = \left\{ \begin{bmatrix} 3s+2t \\ -3s+t \\ s+2t \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$= \left\{ s\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$= \left\{ \vec{b} \in \mathbb{R}^3 : \vec{b} = s\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \text{ for } s,t \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

Because H is a span, H is a space

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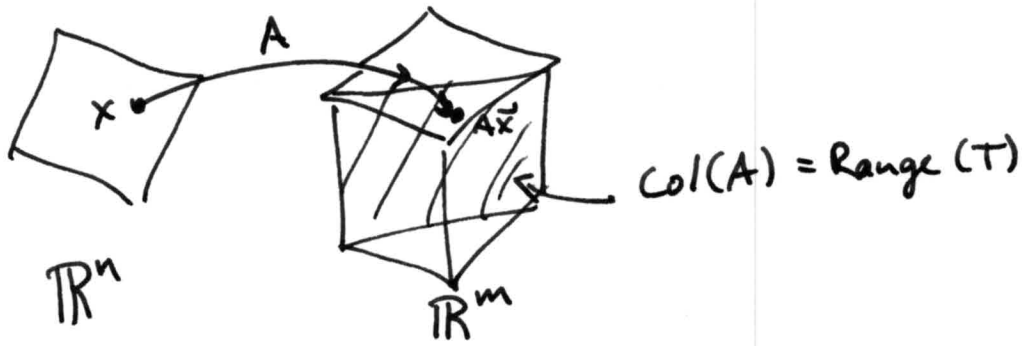
Column Space and Null Space

1. Fix an $m \times n$ matrix $A = \begin{bmatrix} \vec{\mathbf{a}}_1 & \dots & \vec{\mathbf{a}}_n \end{bmatrix}$. Define the Col(A) and Null(A)

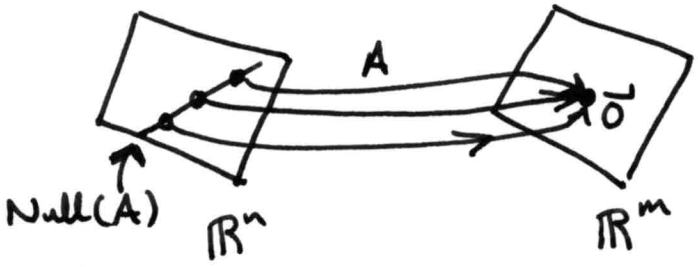
$$\text{Col}(A) = \text{Span}\{\vec{a_1}, \dots, \vec{a_n}\}$$

$$\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$$

2. Draw a sketch that illustrates Col(A) using the linear transformation $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$



3. Draw a sketch that illustrates Null(A) using the linear transformation $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$



4. What is conveyed by the word "space" in Null*space* and Column *space*?

They are both <u>subspaces</u>.

that is: they contain $\vec{0}$
they are closed under adding
they are closed under scaling

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5. Let $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 8 & 10 \\ 1 & 5 & 6 \end{bmatrix}$. Determine if $\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$ is in Col(A) and if $\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$ is in Null(A) .

$\vec{v} \in Col(A) \iff [A|\vec{v}]$ is consistent

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 2 \\ 2 & 8 & 10 & 6 \\ 1 & 5 & 6 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

the system is <u>inconsistent</u>

So $\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} \notin Col(A)$

$\vec{v} \in Null(A) \iff A\vec{v} = \vec{0}$

<u>compute</u>:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 8 & 10 \\ 1 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

<u>conclude</u>

So $\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \in Null(A)$

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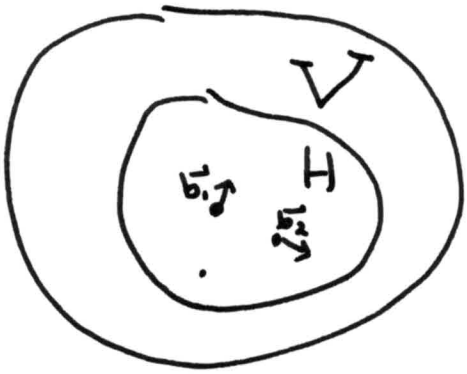
Basis

1. Let H be a subspace of a vector space V, and let $\mathcal{B}$ be a set of vectors in H.

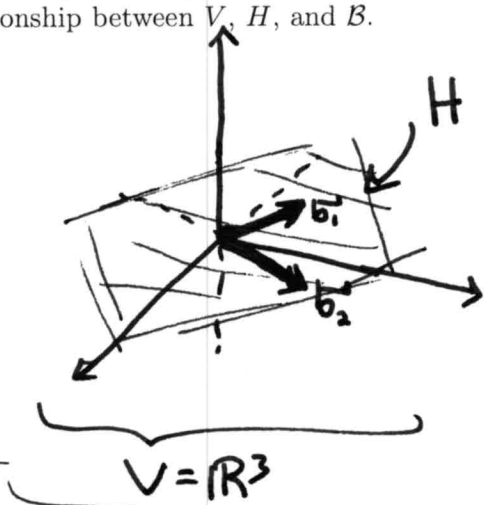
   State the definition of "$\mathcal{B}$ is a basis for H."

$\mathcal{B}$ is a basis for H
if <u>BOTH</u> ① $\mathcal{B}$ is linearly independent
<u>AND</u> ② $H = \text{Span}\,\mathcal{B}$.

2. Suppose that $H \subseteq V$ is a subspace of V, and that $\mathcal{B} = \{b_1, \ldots, b_n\} \subset H$ as in the definition of a basis. Draw a picture that illustrates the relationship between V, H, and $\mathcal{B}$.



<u>OR</u>



3. Let $\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Show that $\mathcal{U}$ is a basis for $\mathbb{R}^3$.

   (In other words: check that it satisfies the definition of a basis).

Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

<u>check</u>
- A has pivot in ea column $\Rightarrow$ its columns are independent
- A has a pivot in each row $\Rightarrow$ its columns span $\mathbb{R}^3$

$\boxed{\text{Conclude: } \{\text{columns of } A\} = \mathcal{U} \text{ are a basis for } \mathbb{R}^3}$

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4. Determine whether the sets below are bases for $\mathbb{R}^3$. Of the sets that are not bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answer.

(a) $\mathcal{U} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 6 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & -2 \\ 4 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

No pivot in col 2 $\Rightarrow$ Not indep

No pivot in row 3 $\Rightarrow$ does not Span $\mathbb{R}^3$

$\mathcal{U}$ is <u>NOT</u> a basis for $\mathbb{R}^3$

(b) $\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 3 & 2 \\ 2 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 \\ 0 & 3 & 2 \\ 0 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Pivot in all columns $\Rightarrow$ independent

pivot in all rows $\Rightarrow$ spans $\mathbb{R}^3$

$\mathcal{U}$ <u>IS</u> a basis for $\mathbb{R}^3$

(c) $\mathcal{U} = \left\{ \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 2 & 0 \\ 6 & 4 \\ 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$$

pivot in each column $\Rightarrow$ <u>IS</u> independent

No pivot in row 3 $\Rightarrow$ does <u>NOT</u> span $\mathbb{R}^3$

$\mathcal{U}$ is <u>NOT</u> a basis for $\mathbb{R}^3$

(d) $\mathcal{U} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} -1 & 2 & 1 & 2 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

already in echelon form

<u>NO</u> pivot in column 4 $\Rightarrow$ <u>NOT</u> independent

pivot in each row $\Rightarrow$ <u>does</u> span $\mathbb{R}^3$

$\mathcal{U}$ is <u>NOT</u> a basis for $\mathbb{R}^3$.

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5. Let $\vec{\mathbf{v}}_1 = \begin{bmatrix}1\\0\\3\\0\end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix}0\\-1\\2\\2\end{bmatrix}, \vec{\mathbf{v}}_3 = \begin{bmatrix}2\\-1\\8\\2\end{bmatrix}, \vec{\mathbf{v}}_4 = \begin{bmatrix}1\\2\\0\\0\end{bmatrix}, \vec{\mathbf{v}}_5 = \begin{bmatrix}2\\0\\7\\4\end{bmatrix}$ and let $H = Span\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4, \vec{\mathbf{v}}_5\}$.

Is $\mathcal{U} = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4, \vec{\mathbf{v}}_5\}$ a basis for H? If not, find a basis $\mathcal{B} \subseteq \mathcal{U}$.

<u>NOTICE:</u> more vectors than rows
$\Rightarrow$ $\mathcal{U}$ is <u>NOT</u> independent
(hence not a basis)

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & 2 \\ 0 & -1 & -1 & 2 & 0 \\ 3 & 2 & 8 & 0 & 7 \\ 0 & 2 & 2 & 0 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 & 2 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 2 & 2 & -3 & 1 \\ 0 & 2 & 2 & 0 & 4 \end{bmatrix} \begin{matrix} \\ \\ r_3 - 3r_1 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 & 2 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 & 4 \end{bmatrix} \begin{matrix} \\ \\ r_3 + 2r_2 \\ r_4 + 2r_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 & 2 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ r_4 - 4r_3 \end{matrix}$$

Pivots in rows 1, 2, 4

$\Rightarrow$

a basis for $\mathcal{U}$ is $\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\3\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\2\\2\end{bmatrix}, \begin{bmatrix}1\\2\\0\\0\end{bmatrix} \right\}$

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7. Let $A = \begin{bmatrix} -2 & 4 & -2 & 4 & 0 \\ 1 & -2 & -1 & 4 & 1 \\ 0 & 0 & 1 & -3 & 1 \\ 3 & -6 & 0 & 3 & -6 \end{bmatrix}$. You are given the fact that $A \sim B = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Find a basis for Col(A) and Null(A).

pivot columns of A form a basis for Col(A)

$$\Rightarrow \mathcal{U} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -6 \end{bmatrix} \right\}$$

is a basis for Col(A)

To find a basis for Null (A) recall that $A \sim B$ implies $[A|\vec{0}] \sim \{B|\vec{0}\}$

so solutions to $A\vec{x} = \vec{0}$ are given by

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Leftrightarrow \begin{cases} x_1 - 2x_2 + x_4 = 0 \\ x_2 \text{ free} \\ x_3 - 3x_4 = 0 \\ x_4 \text{ free} \\ x_5 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 2x_2 - x_4 \\ x_2 \text{ free} \\ x_3 = 3x_4 \\ x_4 \text{ free} \\ x_5 = 0 \end{cases}$$

$$\vec{x} = \begin{bmatrix} 2x_2 - x_4 \\ x_2 + 0 \\ 0 + 3x_4 \\ 0 + x_4 \\ 0 + 0 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$$

$$\text{Null}(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Final Exam Review Math 202

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Coordinate Systems

1. Fix a basis $\mathcal{B} = \left\{ \overset{\vec{b}_1}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}, \overset{\vec{b}_2}{\begin{bmatrix} -1 \\ 3 \end{bmatrix}} \right\}$ for $\mathbb{R}^2$.

if $\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2$ then $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

(a) Find the vector $\vec{\mathbf{v}}$ determined by the coordinate vector $[\vec{\mathbf{v}}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$$\vec{v} = (3)\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

(b) Find the coordinate vector $[\vec{\mathbf{v}}]_{\mathcal{B}}$ of $\vec{\mathbf{v}} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$.

$$\left[\begin{array}{cc|c} 2 & -1 & 4 \\ -1 & 3 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 3 & 4 \\ 2 & -1 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 3 & 4 \\ 0 & 5 & 12 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} -1 & 3 & 4 \\ 0 & 1 & 12/5 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 0 & -16/5 \\ 0 & 1 & 12/5 \end{array}\right]$$

$\frac{20}{5} - \frac{36}{5} = -\frac{16}{5}$

$x_1 = 5$

$x_2 = 3$

$\begin{bmatrix} -16/5 \\ 12/5 \end{bmatrix}$

2. Fix a basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ for $H = Span\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$.

If $\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2$ then $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

(a) Find the vector $\vec{\mathbf{v}}$ determined by the coordinate vector $[\vec{\mathbf{v}}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

$$\vec{v} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

(b) Find the coordinate vector $[\vec{\mathbf{v}}]_{\mathcal{B}}$ of $\vec{\mathbf{v}} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$ relative to $\mathcal{B}$.

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & -2 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

$$[v]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

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If $\vec{v} = c_1\vec{b_1} + c_2\vec{b_2} + c_3\vec{b_3}$

then $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

3. Fix a basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$.

(a) Find the vector $\vec{\mathbf{v}}$ determined by the coordinate vector $[\vec{\mathbf{v}}]_{\mathcal{B}} = \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix}$

$$\vec{v} = (0.5)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (0.5)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1.5 \\ 1 \end{bmatrix}$$

(b) Find the coordinate vector $[\vec{\mathbf{v}}]_{\mathcal{B}}$ of $\vec{\mathbf{v}} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ relative to $\mathcal{B}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \underline{\text{so}} \quad [v]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

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4. Fix a basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \right\}$ for $\mathbb{R}^3$. $\vec{x} = \begin{bmatrix} 0.5 \\ 1 \\ -1 \end{bmatrix}$

Use an inverse matrix to find $[\vec{v}]_{\mathcal{B}}$ for the given $\vec{x}$ and basis $\mathcal{B}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ r_2 - 2r_1 \\ r_3 - r_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \begin{array}{l} \\ \\ r_3 - r_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 3 & -3 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \begin{array}{l} r_1 - 3r_3 \\ r_2 + r_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 3 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \begin{array}{l} r_1 + 2r_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 3 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \begin{array}{l} \\ -r_2 \\ \\ \end{array}$$

$$\underbrace{\begin{bmatrix} -4 & 3 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}}_{B^{-1}}$$

$$[\vec{v}]_{\mathcal{B}} = B^{-1}[\vec{v}] = \begin{bmatrix} -4 & 3 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2+3+1 \\ 0.5+0+1 \\ 0.5+-1+-1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1.5 \\ -1.5 \end{bmatrix}$$

Name: _____ Section: _____

Dimension of a Vector Space

1. Find the dimension of the subspace H of $\mathbb{R}^3$ spanned by $\begin{bmatrix}3\\3\\0\end{bmatrix}$, $\begin{bmatrix}3\\2\\-1\end{bmatrix}$, $\begin{bmatrix}3\\4\\1\end{bmatrix}$, and $\begin{bmatrix}0\\1\\1\end{bmatrix}$.

dimension = size of a basis

pivot columns of A gives a basis for $Col(A)$

$$\begin{bmatrix}3 & 3 & 3 & 0\\3 & 2 & 4 & 1\\0 & -1 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}3 & 3 & 3 & 0\\0 & -1 & 1 & 1\\0 & -1 & 1 & 1\end{bmatrix}$$

$$\sim \begin{bmatrix}3 & 3 & 3 & 0\\0 & -1 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

Pivot cols.

$\mathcal{U} = \left\{\begin{bmatrix}3\\3\\0\end{bmatrix}, \begin{bmatrix}3\\2\\-1\end{bmatrix}\right\}$ is a basis for H

dimension of $H = 2$.

---

2. Find the dimension of the subspace H of $\mathbb{R}^3$ spanned by $\begin{bmatrix}-2\\2\\0\end{bmatrix}$, $\begin{bmatrix}2\\0\\2\end{bmatrix}$, and $\begin{bmatrix}-1\\4\\2\end{bmatrix}$.

pivot columns form a basis for $H = Col(A)$

$$\begin{bmatrix}-2 & 2 & -1\\2 & 0 & 4\\0 & 2 & 2\end{bmatrix} \sim \begin{bmatrix}-2 & 2 & -1\\0 & 2 & 3\\0 & 2 & 2\end{bmatrix}$$

$$\sim \begin{bmatrix}-2 & 2 & -1\\0 & 2 & 3\\0 & 0 & -1\end{bmatrix}$$

$\mathcal{U} = \left\{\begin{bmatrix}-2\\2\\0\end{bmatrix}, \begin{bmatrix}2\\0\\2\end{bmatrix}, \begin{bmatrix}-1\\4\\2\end{bmatrix}\right\}$

is a basis for $Col(A) = H$

dimension of $H = 3$

Name: _____ Section: _____

3. Find a basis and state the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a+b \\ a-b \\ a \end{bmatrix} \in \mathbb{R}^3 : a, b \in \mathbb{R} \right\}$$

$$H = \left\{ a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} : a,b \in \mathbb{R} \right\}$$

$$= \cdots = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$ ← these vectors <u>Span</u> H.

<u>check</u>: are they <u>independent</u>?

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$ ← they <u>are</u> independent.

$\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ is a basis for H $\Rightarrow$ $\boxed{\text{dimension of } H = 2}$

4. Find a basis and state the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a+2b+c \\ b+c \\ a-c \end{bmatrix} \in \mathbb{R}^3 : a, b, c \in \mathbb{R} \right\}$$

$$H = \left\{ a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} : a,b,c \in \mathbb{R} \right\} = \cdots = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

these vectors <u>Span</u> H.

<u>check</u>: are they independent?

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ $\Rightarrow$ <u>not</u> independent.

Pick pivot col's

$\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for H

$\Rightarrow$ $\boxed{\text{Dimension of } H = 2}$.

Name: _____ Section: _____

5. Define the dimension of a vector space V.

The dimension of V is the size of one/any basis of V

6. Fill in the blanks in the following

Theorem: Fix a vector space $V \neq \{\emptyset\}$ and suppose $V = \operatorname{Span} \mathcal{S}$

If $\mathcal{S}$ is *not* a basis, then $\mathcal{S}$ can be refined to give a basis $\mathcal{B} \subseteq \mathcal{S}$ of V

Theorem: Let V be a vector space with basis $\mathcal{B} = \{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n\}$.

Then *every* basis of V contains exactly n vectors.

7. State *The Basis Theorem*

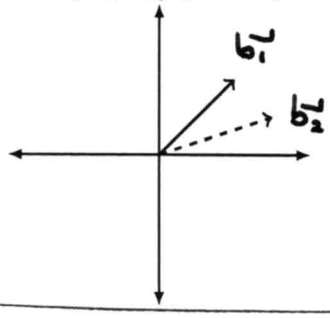
Theorem: Let V be a vector space with basis $\mathcal{B} = \{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n\}$.

Then *every* linearly indep $\mathcal{U}$ with exactly n vectors is a Basis of V.

Name: _____ Section: _____

8. Determine if each of the following set of vectors forms a basis for $\mathbb{R}^2$. You must **justify your answer.**

(a) $\mathcal{U} = \{\vec{b}_1, \vec{b}_2\}$ given by



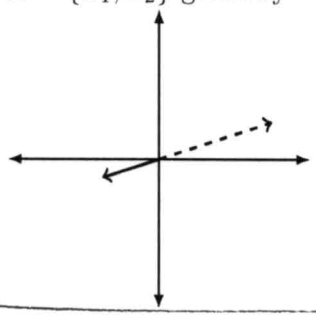
$\mathcal{U} \subseteq \mathbb{R}^2$

① is independent

② $|\mathcal{U}| = 2 = \dim(\mathbb{R}^2)$

$\Rightarrow \mathcal{U}$ is a basis of $\mathbb{R}^2$ by the basis theorem

(b) $\mathcal{U} = \{\vec{b}_1, \vec{b}_2\}$ given by

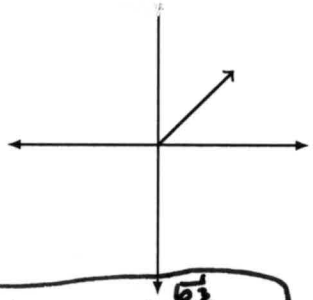


$\mathcal{U} \subseteq \mathbb{R}^2$

① is NOT independent

$\Rightarrow \mathcal{U}$ is NOT a basis of $\mathbb{R}^2$

(c) $\mathcal{U} = \{\vec{b}_1$ given by



$\mathcal{U} \subseteq \mathbb{R}^2$

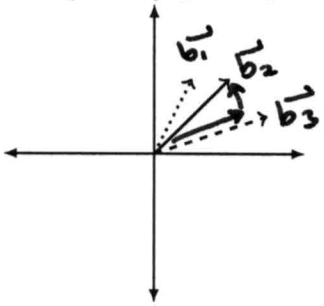
① is independent

② $|\mathcal{U}| = 1 \neq 2 = \dim(\mathbb{R}^2)$

$\Rightarrow \mathcal{U}$ is NOT a basis of $\mathbb{R}^2$

(each basis contains 2 vectors)

(d) $\mathcal{U} = \{\vec{b}_1, \vec{b}_2\}$ given by



$\mathcal{U} \subseteq \mathbb{R}^2$

① is NOT independent

NOTE: $\vec{b_1} \in \text{Span}\{\vec{b_1}, \vec{b_3}\}$

$\Rightarrow \mathcal{U}$ is NOT a basis of $\mathbb{R}^2$

Name: _____ Section: _____

Rank + Nullity

1. Determine the dimension of the $Null(A)$ and $Col(A)$ for the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 3 & 6 & 2 & 4 & 2 \\ -1 & -2 & -2 & 0 & 2 \end{bmatrix} \rightarrow \begin{array}{l} Col(A) \subseteq \mathbb{R}^3 \\ Null(A) \subseteq \mathbb{R}^5 \end{array}$$

Find j and k s.t. $Col(A) \subseteq \mathbb{R}^j$ and $Null(A) \subseteq \mathbb{R}^k$. Then find a basis for $Col(A)$ and $Null(A)$.

Solve $A\vec{x} = \vec{0}$ for basis of $Null(A)$

$$[A|\vec{0}] = \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 6 & 2 & 4 & 2 & 0 \\ -1 & -2 & -2 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & -1 & 1 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 2 & 0 \\ 0 & 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} x_1 + 2x_2 + 2x_4 + 2x_5 = 0 \\ x_2 \text{ free} \\ x_3 - x_4 + 2x_5 = 0 \\ x_4, x_5 \text{ free} \end{cases}$$

<u>Solutions</u> are

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 2x_4 - 2x_5 \\ x_2 + 0 + 0 \\ 0 + x_4 - 2x_5 \\ 0 + x_4 + 0 \\ 0 + 0 + x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

a Basis for $Null(A)$

$$\mathcal{U} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow \boxed{\text{Nullity of } A = \dim(Null(A)) = 3}$$

Pivot columns of A form a basis of $Col(A)$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \right\} \Rightarrow \boxed{\text{Rank of } A = \dim(Col(A)) = 2}$$

Name: _____ Section: _____

2. If the null space of a 8×7 matrix A is 5-dimensional, what is the dimension of $Col(A)$?

m n

$n = 7$

Nullity = dim (Null(A)) = 5

rank + Nullity = n

rank + 5 = 7

rank = 2

dim (Col(A)) = 2

3. If the column space of a 8×7 matrix A is 2-dimensional, what is the dimension of $Null(A)$?

m n

$n = 7$

rank = dim(Col(A)) = 2

rank + Nullity = n

2 + Nullity = 7

Nullity = 5

dim (Null(A)) = nullity = 5

Name: _____ Section: _____

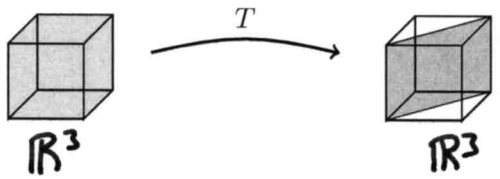
9. For each example below, suppose that T is a linear transformation with standard matrix A, which has the domain, co-domain, and range indicated by the corresponding sketch.

(1) Find the Rank and Nullity of A, (2) find j and k so that $Col(A) \subseteq \mathbb{R}^j$ and $Null(A) \subseteq \mathbb{R}^k$, and (3) indicate the dimension of each space.

(a)

$Col(A) \subseteq \mathbb{R}^3$

$Null(A) \subseteq \mathbb{R}^3$



$n = 3$

$col(A) = range(T)$ is 2 dimensional

$\Rightarrow$ rank $= 2$

rank + Nullity $= n$

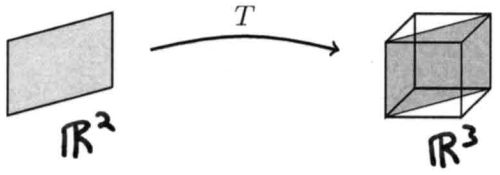
2 + Nullity $= 3$

Nullity $= 1 \Rightarrow \dim(Null(A)) = 1$

(b)

$Col(A) \subseteq \mathbb{R}^3$

$Null(A) \subseteq \mathbb{R}^2$



$n = 2$

$col(A) = range(T)$ is 2 dimensional

$\Rightarrow$ rank $= 2$

rank + Nullity $= n$

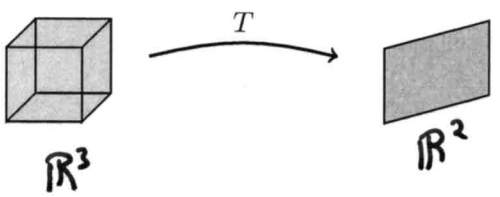
2 + Nullity $= 2$

Nullity $= 0 \Rightarrow \dim(Null(A)) = 0$

(c)

$Col(A) \subseteq \mathbb{R}^3$

$Null(A) \subseteq \mathbb{R}^3$



$n = 3$

$Col(A) = range(T)$ is 2 dimensional

$\Rightarrow$ rank $= 2$

rank + Nullity $= n$

2 + Nullity $= 3$

Nullity $= 1 \Rightarrow \dim(Null(A)) = 1$

Name: _____ Section: _____

Eigenvalues and Eigenvectors

1. Let $A = \begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}$, let $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Are $\vec{v}$ and $\vec{\mathbf{u}}$ eigenvectors of A? If so, what is the eigenvalue?

   Justify your answer.

$$A\vec{v} = \begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (1)\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is an eigenvector w/ eigenvalue $= \lambda$

$$A\vec{u} = \begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix} \neq \lambda \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for any } \lambda$$

is not an eigenvector

2. Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. Is $\lambda = 2$ an eigenvalue of A? If it is, find all the corresponding eigenvectors.

   Justify your answer.

$\lambda = 2$ is an eigenvalue

$\Leftrightarrow$

$A\vec{x} = 2\vec{x}$ has nontrivial sln

$\Leftrightarrow$

$(A - 2I_2)\vec{x} = \vec{0}$ has nontrivial sln

$$A - 2I_2 = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

$$[A - 2I_2 \mid \vec{0}] = \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 - x_2 = 0 \\ x_2 \text{ free} \end{cases} \Leftrightarrow \begin{cases} x_1 = x_2 \\ x_2 \text{ free} \end{cases}$$

no pivot $\Rightarrow \infty$ many slns $\Rightarrow \lambda = 2$ is an eigenvalue

eigenvectors are the solutions

$$\vec{x} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with $x_2 \neq 0$

Name: _____ Section: _____

3. Let $A = \begin{bmatrix} -5 & 2 & -4 \\ -4 & 1 & -4 \\ 4 & -2 & 3 \end{bmatrix}$, with $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Are $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ eigenvectors of A?

If so, what are the eigenvalues. Justify your answer.

$$A\vec{u} = \begin{bmatrix} -5 & 2 & -4 \\ -4 & 1 & -4 \\ 4 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

is an eigenvector w/ eigenvalue $\lambda = -1$

$$A\vec{v} = \begin{bmatrix} -5 & 2 & -4 \\ -4 & 1 & -4 \\ 4 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

is an eigenvector w/ eigenvalue $\lambda = -1$

4. Let $A = \begin{bmatrix} -2 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & -2 \end{bmatrix}$. Is $\lambda = -2$ an eigenvalue of A? If it is, find all the corresponding eigenvectors.

Justify your answer.

$\lambda = -2$ is an eigenvalue
$\Leftrightarrow A\vec{x} = \lambda\vec{x}$ has nontrivial solution
$\Leftrightarrow (A - \lambda I_3)\vec{x} = \vec{0}$ has nontrivial solution

$$A - \overset{(-2)}{I_3} = \begin{bmatrix} -2 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

$$[A - (-2)I_3 | \vec{0}] = \left[\begin{array}{ccc|c} 0 & -3 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Leftrightarrow \begin{cases} x_1 \text{ free} \\ x_2 = 0 \\ x_3 \text{ free} \end{cases} \Leftrightarrow \vec{x} = \begin{bmatrix} x_1 + 0 \\ 0 + 0 \\ 0 + x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

eigenvector space of $A = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

Name: _____ Section: _____

5. State the definition of the characteristic polynomial of an $n \times n$ matrix A.

the characteristic polynomial of A is $\det(A - \lambda I_n)$

6. Prove that λ is an eigenvalue of A if and only if it is a root of the characteristic polynomial of A.

<u>by rephrasing:</u>

λ is an eigenvalue of A

$\Longleftrightarrow$

$A\vec{x} = \lambda\vec{x}$ has a <u>nontrivial</u> sln.

$\Longleftrightarrow$

$(A - \lambda I_n)\vec{x} = \vec{0}$ has a nontrivial sln

$\Longleftrightarrow$

$A - \lambda I_n$ lacks a pivot in some column

$\Longleftrightarrow$

$A - \lambda I_n$ is <u>not</u> invertible

$\Longleftrightarrow$

$\det(A - \lambda I_n) = 0.$

Name: _____ Section: _____

6. Find the eigenvalues of the matrix $A = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$.

λ is an eigenvalue of A

$\iff$

$\det(A - \lambda I_2) = 0$

$$A - \lambda I_2 = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_2) = (5-\lambda)(-1-\lambda) - 4(-2)$$
$$= -5 + \lambda - 5\lambda + \lambda^2 + 8$$
$$= \lambda^2 - 4\lambda + 3$$
$$= (\lambda - 3)(\lambda - 1)$$

eigenvalues are $\lambda = 3$ and $\lambda = 1$

7. Find the characteristic polynomial and the real eigenvalues of the following matrices.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & -4 \\ 2 & 1 & 0 \end{bmatrix}$$

$$A - \lambda I_3 = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & -4 \\ 2 & 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 & 2 \\ 2 & -1-\lambda & -4 \\ 2 & 1 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I_3) = \det \begin{vmatrix} -\lambda & 1 & 2 \\ 2 & -1-\lambda & -4 \\ 2 & 1 & -\lambda \end{vmatrix} = (-\lambda) \begin{vmatrix} -1-\lambda & -4 \\ 1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 2 & -4 \\ 2 & -\lambda \end{vmatrix} + 2 \begin{vmatrix} 2 & -1-\lambda \\ 2 & 1 \end{vmatrix}$$

$$= (-\lambda)\Big((-1-\lambda)(-\lambda) - (-4)(1)\Big) - 1\Big(-2\lambda - 2(-4)\Big) + 2\Big(2\cdot 1 - 2(-1-\lambda)\Big)$$

$$= -\lambda(\lambda + \lambda^2 + 4) + 2\lambda - 8 + 2(2 + 2 + 2\lambda)$$

$$= -\lambda^2 - \lambda^3 - 4\lambda + 2\lambda \underline{- 8 + 8} + 4\lambda$$

$$= -\lambda^3 - \lambda^2 + 2\lambda = -\lambda(\lambda^2 + \lambda - 2) = \lambda(\lambda + 2)(\lambda - 1)$$

real eigenvalues are $\lambda = 0$, $\lambda = -2$, & $\lambda = 1$.

Name: _____ Section: _____

8. Let $A = \begin{bmatrix} -3 & 10 \\ -2 & 6 \end{bmatrix}$

Find all eigenvalues of A, then find the eigenvectors that correspond to each eigenvalue.

$$A - \lambda I_2 = \begin{bmatrix} -3-\lambda & 10 \\ -2 & 6-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_2) = \det \begin{bmatrix} -3-\lambda & 10 \\ -2 & 6-\lambda \end{bmatrix}$$

$$= (-3-\lambda)(6-\lambda) - 10(-2)$$

$$= -18 - 6\lambda + 3\lambda + \lambda^2 + 20$$

$$= \lambda^2 - 3\lambda + 2$$

$$= (\lambda - 2)(\lambda - 1)$$

eigenvalues are $\lambda = 2$ & $\lambda = 1$

corresponding to $\lambda = 2$

$$A - 2I_2 = \begin{bmatrix} -3 & 10 \\ -2 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 10 \\ -2 & 4 \end{bmatrix}$$

$$[A - 2I_2 | \vec{0}] = \left[\begin{array}{cc|c} -5 & 10 & 0 \\ -2 & 4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 - 2x_2 = 0 \\ x_2 \text{ free} \end{cases} \qquad x_1 = 2x_2$$

$$\vec{x} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ for } x_2 \neq 0$$

corresponding to $\lambda = 1$

$$A - (1)I_2 = \begin{bmatrix} -3 & 10 \\ -2 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 10 \\ -2 & 5 \end{bmatrix}$$

$$[A - (1)I_2 | \vec{0}] = \left[\begin{array}{cc|c} -4 & 10 & 0 \\ -2 & 5 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 2 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} 2x_1 - 5x_2 = 0 \\ x_2 \text{ free} \end{cases} \qquad x_1 = \frac{5}{2}x_2$$

$$\vec{x} = \begin{bmatrix} 5/2 \cdot x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 5/2 \\ 1 \end{bmatrix} \text{ for } x_2 \neq 0.$$

Name: _____ Section: _____

10. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & -2 \end{bmatrix}$

Find all eigenvalues of A, then find the eigenvectors that correspond to each eigenvalue.

λ is an eigenvalue $\Leftrightarrow$ $\det(A - \lambda I_3) = 0$

$$A - \lambda I_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ -1 & 0 & (-2-\lambda) \end{bmatrix}$$

$$\det(A - \lambda I_3) = -\lambda \begin{vmatrix} -\lambda & 0 \\ 0 & (-2-\lambda) \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 0 \\ -1 & (-2-\lambda) \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & -\lambda \\ -1 & 0 \end{vmatrix}$$

$$= -\lambda\left((-\lambda)(-2-\lambda) - 0\right) - 0 + 1\left(0 - (-\lambda)(-1)\right)$$

$$= -2\lambda^2 - \lambda^3 - \lambda$$

$$= -\lambda^3 - 2\lambda^2 - \lambda = -\lambda(\lambda^2 + 2\lambda + 1)$$

$$= -\lambda(\lambda+1)(\lambda+1)$$

eigenvalues of A are $\lambda = 0$ & $\lambda = -1$

eigen vectors:

corresponding to $\lambda = 0$

$$[A - 0 I_3 | \vec{0}] = \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 \text{ free} \\ x_3 = 0 \end{cases}$$

$$\vec{x} = \begin{bmatrix} 0 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ for } x_3 \neq 0.$$

Corresponding to $\lambda = -1$

$$[A - (-1) I_3 | \vec{0}] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Leftrightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \\ x_3 \text{ free} \end{cases} \Leftrightarrow \begin{cases} x_1 = -x_3 \\ x_2 = x_3 \\ x_3 \text{ free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

for $x_3 \neq 0$.

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9. Let $A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$

Find all eigenvalues of A, then find the eigenvectors that correspond to each eigenvalue.

$$\det(A-\lambda I_3) = \det\begin{pmatrix} -1-\lambda & 2 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{pmatrix}$$

$$= (-1-\lambda)\begin{vmatrix} (1-\lambda) & 0 \\ 2 & -1-\lambda \end{vmatrix} - 2\underbrace{\begin{vmatrix} 0 & 0 \\ 0 & -1-\lambda \end{vmatrix}}_{0} + 0\cdot\underbrace{\begin{vmatrix} 0 & 1-\lambda \\ 0 & 2 \end{vmatrix}}_{0}$$

$$= (-1-\lambda)\big((1-\lambda)(-1-\lambda) - 0\big) + 0 + 0$$

$$= (-1)^3(\lambda+1)(\lambda-1)(\lambda+1) = (-1)(\lambda+1)^2(\lambda-1)$$

eigenvalues are when $\lambda=-1$ and $\lambda=1$

corresponding to $\lambda=-1$

$$[\underbrace{A-(-1)I_3}_{\begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}+\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}|\vec{0}] = \left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Leftrightarrow \begin{cases} x_1 \text{ free} \\ x_2 = 0 \\ x_3 \text{ free} \end{cases}$$

eigenvectors are

$$\vec{x} = \begin{bmatrix} x_1 \\ 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

with x_1, x_2 not both zero

corresponding to $\lambda=1$

$$A-(1)I_3 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A-(1)I_3|\vec{0}] = \left[\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Leftrightarrow \begin{cases} x_1 - x_3 = 0 & x_1 = x_3 \\ x_2 - x_3 = 0 & x_2 = x_3 \\ x_3 \text{ free} \end{cases}$$

eigenvectors are

$$\vec{x} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ with } x_3 \neq 0$$

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Final Exam Review

1. Study Exam 1 Review Packet
2. Study Exam 2 Review Packet
3. Study Exam 3 Review Packet
4. Study Material since Exam 3